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SHOCKLESS COMPRESSION OF A BAROTROPIC GAS†

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A problem concerning the shockless "cold" compression of one-dimensional layers (planar, cylindrical and spherical) of a barotropic gas which requires a minimum amount of external energy in order to attain the specified degree of compression is formulated and solved. The initial state of gas is assumed to be homogeneous. In the planar case, an exact solution of the problem is obtained (laws for the optimal control of the motion of the piston are constructed) using Pontryagin's maximum principle while, in the cylindrical and spherical cases, an approximate solution is obtained using the method of characteristic series. In the planar case, the magnitude of the energy gain is found compared with the traditional self-similar method of compression which had turned out to be quite appreciable and to depend on the equation of state. The results of numerical calculations are presented for the cylindrical case which was studied in greater detail. These calculations were carried out on the basis of the analytically constructed law for the optimal control of the motion of the piston with a single point of control commutation. A brief account of some of the results is given in [1].

IT HAD already been shown by Rayleigh and Hugoniot [2] that, by using a class of self-similar Riemann waves, it is possible, in the isentropic compression of a planar layer of a polytropic gas, to obtain a gas density which may be as large as desired. The possibility of the unlimited compression of a gaseous cylinder and a gaseous sphere has been established [3, 4] using classes of self-similar cylindrical and spherical flows which have been studied in detail [5] (in the case of problems of the displacement of a gas). It was pointed out in [3, 4] that the processes involved in the shockless compression of a gas are favourable energy-wise since they do not lead to a large increase in the kinetic energy and to the pronounced heating of the substance which is observed during shock compression. Such processes can therefore play an important role in the realization of laser thermonuclear synthesis when the compression of the targets is achieved using a special form of shaped-pulse laser radiation.

A more general problem (compared with [2–4]) of the shockless compression of layers of a barotropic gas by means of a piston up to an arbitrary final average density with the minimum energy requirements for the motion of the piston is considered below. In the case of a cylindrical or spherical symmetry of the layer, there is not longer a solution of this problem in the class of self-similar motions and wide classes of flows have to be invoked.

1. At the initial instant of time $t = 0$, let a homogeneous layer of an immobile gas $p = p_0$, $\rho = \rho_0$, $u = 0$ with the equation of state $p = p(\rho)$ (the conventional notation) be located between the surfaces $\xi = R_f$ and $\xi = R_0$, $R_0 > R_f \geqslant 0$, where ξ is a radial or planar coordinate. We shall assume

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that the units of measurements are chosen such that $c_0^2 = p'(\rho_0) = 1$, where c_0 is the initial velocity of sound, and the compression process is realized in the time interval $t \in [0, t_k)$, where $t_k = R_0 - R_f$ corresponds to the time taken by the sonic perturbation to traverse the layer. The surface $\xi = R_f$ acts as a fixed wall and $\xi = R_0$ corresponds to the initial position of the piston R_t.

We shall seek the law of motion of the piston $R_t \xi = f(t) f(0) = R_0, f'(t) \leq 0$ such that:

1. $f(t_k) = R_k$, $R_0 > R_k > R_f$, where R_k is a specified quantity which characterizes the degree of compression of the layer.
2. When $t \in [0, t_k)$, the flow of the gas is shockless.
3. $E(f(t_k))$, the work done by external forces in displacing the piston should be a minimum.

2. Let us initially consider the planar case and assume the $R_f \geq 0$ and $p(\rho) \geq c_0^2$. The equations of the isentropic one-dimensional motions of the gas have the form

$$u_t + uu_\xi + \rho^{-1} p'(\rho) \rho_\xi = 0 \qquad \rho_t + (u\rho)_\xi + N\rho u \xi^{-1} = 0 \tag{2.1}$$

where the values $N = 0, 1, 2$ correspond to the planar, cylindrical and spherical cases.

It is clear that, in the planar case ($N = 0$), starting from $t = 0$ the weak disturbance from the piston propagates into the immobile gas at a single velocity while the perturbed flow between the weak disturbance and the piston is a travelling Riemann wave and is described by the relationships [2]

$$u + \int_{\rho_0}^{\rho} c\rho^{-1} d\rho = 0, \quad c^2 = p'(\rho) \tag{2.2}$$

$$\xi = (u - c)t + \Psi(u) \tag{2.3}$$

with an arbitrary function $\Psi(u)$.

Let us first find the law of motion of the piston R_t^0 which ensures an unbounded shockless compression of the planar layer at the instant of time $t = t_k$. By exactly the same reasoning [2] the function $\Psi(u)$, which ensures the intersection of all the rectilinear characteristics starting from the trajectory of the motion of the piston, that is, of the curve DCA at the point $A(R_k, t_k)$ of the ξ, t plane (Fig. 1), will correspond to such a law. Here, $f'(0) = 0$ since, if $f'(0) < 0$, a shock wave will immediately be formed in the flow. We shall assume that $R_t^0 \rho = g(t)$, $g(0) = \rho_0$ on the piston and that the no-passage condition $u = f'(t)$ is satisfied.

The equations of the rectilinear characteristics passing through the point A have the form

$$\xi - R_f = (u - c)(t - t_k) \tag{2.4}$$

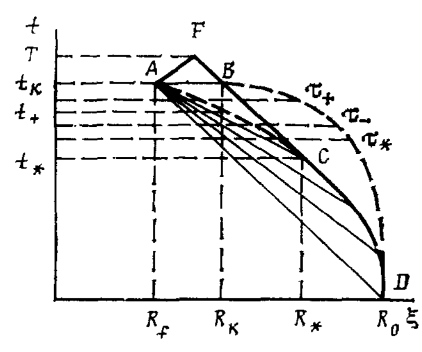

FIG. 1.

Hence, the function $\Psi(u)$ is

$$\Psi(u) = R_f - (u - c)t_k$$

and, along the line of the piston, we obtain two relationships from (2.2) and (2.4) for finding the functions $f(t)$ and $g(t)$

$$f'(t) + \int_{\rho_*}^{g(t)} \sqrt{p'(g(t))}\, g^{-1}(t)\, g'(t)\, dt = 0$$
$$f(t) - R_f = (f'(t) - \sqrt{p'(g(t))})(t - t_k) \tag{2.5}$$

By eliminating the function $f(t)$ from relationships (2.5) and integrating the resulting differential equation for $g(t)$, we find the following integral:

$$\sqrt{p'(g(t))}\, g(t)(t - t_k) = -\rho_0 t_k \tag{2.6}$$

from which the density of the gas $g(t)$ on the piston is determined.

By integrating the equation for $f(t)$, we get

$$f(t) = R_0 - t - \rho_0 t_k (t - t_k) \int_{-t_k}^{t-t_k} z^{-2} g^{-1}(z + t_k)\, dz = f_*(t) \tag{2.7}$$

Here, the limit of the factor accompanying $\rho_0 t_k$ when $t \to t_k$ is equal to $\lim_{t\to t_k}[-g^{-1}(t)] = 0$ since it follows from (2.6) that $g(t) \to \infty$ when $t \to t_k$.

Hence, the dependence (2.7) determines the law of motion of the piston R_t^0 which ensures the unbounded compression of a planar layer of a barotropic gas at the instant $t = t_k$. In the case of a polytropic gas with an equation of state $p = a^2\rho^\gamma$ ($\gamma > 1$ is the adiabatic index, $a^2 = \gamma^1 \rho_0^{1-\gamma}$ and $p_0 = \gamma^1 \rho_0$, where ρ_0 and p_0 are the background values of the density and pressure), we obtain from (2.7) [2] that

$$f(t) = \frac{2}{\gamma - 1}(t - t_k) + R_f + \frac{\gamma + 1}{\gamma - 1} t_k \left(1 - \frac{t}{t_k}\right)^{2/(\gamma+1)} \tag{2.8}$$

The energy required for the motion of the piston R_t in the time interval $[0, t_k)$ is represented by the integral

$$E(f(t_k)) = -\int_0^{t_k} p(q(f')) f'(t)\, dt \tag{2.9}$$

where the function $q(f')$ is defined implicitly by relationship (2.2) when $u = f'(t)$.

If we consider the problem of minimizing the functional (2.9) in the class of functions $f(t)$ which satisfy the specified boundary condition $f(0) = R_0$, $f(t_k) = R_k$ then the linear function $f(t) = (R_k - R_0)t/t_k$ such that $f'(0) \neq 0$ will be the solution of the standard variational problem which arises. Shockless compression accompanying such a motion of the piston R_t is impossible and it is therefore necessary to restrict the class of permissible functions $v = f'(t) \in V$ in order to ensure that there are no shock waves.

We shall next consider a class of equations of state for which the velocity of sound c is a nondecaying function of ρ. It then follows from (2.2) that, as $|u|$ on the piston increases, the magnitude of $c(u)$ does not decrease and the slope of the characteristics (2.4), which is determined by the magnitude of $|u - c|$, also increases. In this case, the class of permissible controls V will consist of the functions v which satisfy the inequalities

$$f_*'(t) \leqslant v \leqslant 0 \tag{2.10}$$

Let us prove analytically in the case of a polytropic gas that, if the inequality

$$F'(t) < f'_*(t) = \frac{1}{\gamma-1}\left[1-\left(1-\frac{t}{t_k}\right)^{-(\gamma-1)/(\gamma+1)}\right] = \zeta(t) \leqslant 0 \tag{2.11}$$

is satisfied in a certain interval of time $[\tau_-, \tau_+]$ (Fig. 1) from the interval $[0, t_k)$ in the case of a control law $\xi = F(t)$ with a piston R_F, then a gradient catastrophe necessarily occurs in the flow of a gas which is determined by the motion R_F at a certain instant $t < t_k$ and a shock wave occurs.

We will write the equation of the rectilinear characteristic (2.4) which starts out from the piston R_F when $t = t_0$ in the form

$$\xi = \left(-1 + \frac{\gamma+1}{2} F'(t_0)\right)(t - t_0) + F(t_0) \tag{2.12}$$

We will find the envelope of the family of characteristics which depend on the parameter t_0 from the relationship which is obtained by differentiating (2.12) with respect to t_0

$$t = t_0 - \frac{2}{\gamma+1}\left(1 - \frac{\gamma-1}{2} F'(t_0)\right)(F''(t_0))^{-1} \tag{2.13}$$

(when the acceleration of the piston R_F does not vanish). Equations (2.12) and (2.13) define the envelope. We note that, in the case of the piston R_t^0, all of the characteristics intersect at $t = t_k$.

As t_0, let us consider the instant τ_* (Fig. 1) at which the relationships

$$f_*'(\tau_*) = F'(\tau_*) < 0, \quad F''(\tau_*) < f_*''(\tau_*) < 0$$

are satisfied, that is, at the instant τ_* the velocities of the pistons R_t^0 and R_F were equalized while the absolute value of the acceleration of the piston R_F was greater than the analogous value in the case of the piston R_t^0. It is clear that, by virtue of (2.11), such an instant where $\tau_* < \tau_-$ is found. Then, by calculating, at $t = \tau_*$, the difference in the times $\Delta t = t_k - t_F$ which correspond to the occurrence of a gradient catastrophe, we get

$$\Delta t = \frac{2}{\gamma+1}\left(\frac{\gamma-1}{2} f_*'(\tau_*) - 1\right)\frac{F''(\tau_*) - f_*''(\tau_*)}{F''(\tau_*) f_*''(\tau_*)} > 0$$

that is, $t_F < t_k$ and a shock wave appears in the flow of the gas caused by the motion of the piston R_F up to the instant t_k.

In the general case of barotropic gas, subject to the assumption which has been made regarding the monotonicity of the increase in the velocity of sound, simpler geometric considerations also confirm that a shock wave is formed in the gas flow at $t < t_k$ when the left-hand side of inequality (2.10) breaks down.

3. The problem of minimizing the functional (2.9) in the class of functions $f(t)$ which satisfy the specified boundary conditions and the constraints (2.10) is a standard problem of optimal control [6]. We shall use Pontryagin's maximum principle to solve it. By putting $p(q(f')) = P(v)$ in (2.9), we get that the optimal control $v(t)$ is found from the condition for a minimum of the function

$$\min_{v \in V} G(v, \lambda), \quad G(v, \lambda) = -P(v)\, v - \lambda v \tag{3.1}$$

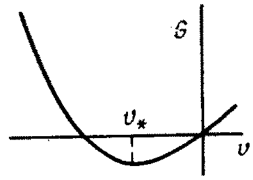

FIG. 2.

where $\lambda < -p_0 < 0$ is a parameter which is found from the condition that the piston R_t passes through the point $t = t_k$, $\xi = R_k$. The qualitative form of the dependence $G(\nu, \lambda)$ at a fixed $\lambda\,(G_\nu(0,\lambda) = -p^0 - \lambda > 0,\ G(-\infty, \lambda) = \infty)$ is shown in Fig. 2.

We note that, when $\lambda > -p_0$, it is impossible to find a trajectory of the motion of the piston R_t which passes through the points D and B (Fig. 1).

It follows from the form of $G(\nu, \lambda)$ that the optimal control ν is formed by continuous splicing of two functions: the function ν is initially identical to $\zeta(t)$ from (2.7) until the function G reaches a minimum at $\nu = \nu_*$ such that, at the instant of time $t = t_*$ (point C in Fig. 1 corresponds to this), $\nu_* = \zeta(t_*)$ and subsequently, at $t_* < t \leq t_k$, the piston moves at a constant velocity $\nu = \nu_*$, that is, the line CB is a straight line. Along CB, the velocity and density of the gas are constant and these values are also kept constant at all points of the triangle ABC such that the final state of the gas is homogeneous at $t = t_k$ on the segment AB (apart from at point A, where $\rho = r_0$, $u = 0$). Hence, uniform compression of the gas up to the specified density is achieved after the instant t_*.

In the case of a polytropic gas, the instant t_* is calculated in an exceedingly simple manner from geometric considerations so that the final law of optimal control has the form

$$
\begin{gathered}
v_0(t) = \zeta(t) \quad 0 \leqslant t \leqslant t_* = t_k\left(1 - s^{(\gamma+1)/2}\right) \\
v_0(t) = \zeta(t_*) = \frac{2}{\gamma - 1}\left(1 - s^{-(\gamma-1)/2}\right), \quad s = \frac{R_k - R_f}{R_0 - R_f}, \quad t_* \leqslant t \leqslant t_k
\end{gathered}
\tag{3.2}
$$

where s is the specified degree of compression of the planar layer.

Let us now establish in the case of a polytropic gas that the sufficient conditions for a minimum of the functional $E(f)$ (2.9) to exist are satisfied when there are small perturbations $h(t)$ of the optimal control law $\nu_0(t)$ from (3.2). In the given case the functional $E(f)$ has the form

$$
E(f) = J(f') = -p_0 \int_0^{t_k} \left(1 - \frac{\gamma - 1}{2} f'\right)^{2\gamma/(\gamma-1)} f'\, dt \tag{3.3}
$$

Let the perturbation $h(t)$ of the law (3.2) be such that the conditions

$$
\zeta(t) \leqslant v_0(t) + h'(t), \quad h(t_k) = h(0) = 0
$$

$$
h'(t) \geqslant \text{ when } 0 \leqslant t \leqslant t_*.
$$

Then, using standard procedures [6], we get

$$
\Delta = J(v_0 + h') - J(v_0) = p_0 \int_0^{t_*} h v_0' R(v_0)\, dt + \int_0^{t_k} O(h^2)\, dt
$$

$$
v_0' = -\frac{2}{(\gamma + 1) t_k}\left(1 - \frac{t}{t_k}\right)^{-2\gamma/(\gamma+1)} < 0
$$

$$
R(v_0) = \gamma\left(1 - \frac{\gamma - 1}{2} v_0\right)^{2/(\gamma+1)}\left(-2 + \frac{3\gamma - 1}{2} v_0\right) < 0
$$

It is clear that, for sufficiently small $h'(t)$, we have $\Delta > 0$ and the conditions for a local minimum to exist are satisfied.

The energy of the piston $J(v_0)$ goes into increasing the internal energy $\Delta\varepsilon$ and the kinetic energy $\Delta\omega$ of the gas layer

$$\Delta\varepsilon = \frac{p_0 t_k}{\gamma - 1}(s^{1-\gamma} - 1), \quad \Delta\omega = \frac{2\gamma p_0 t_k}{\gamma - 1}(s^{(1-\gamma)/2} - 1)^2 \tag{3.4}$$

It follows from (3.4) that, at low compression, when $s \to 1$, the ratio $\Delta\varepsilon/\Delta\omega$ increases in an unbounded manner, that is, most of the piston energy goes into increasing the internal energy while, at high compression when $s \to 0$, this ratio tends to a finite limit.

The estimation of the gain in energy E_0 which is expended in the case of the optimal law of motion of the piston (3.2) compared with traditional control methods and, in particular, compared with the energy E_s of the fastest possible compression which is expended at the instant t_+ (Fig. 1) when the trajectory of the piston (2.8) arrives at the point $\xi = R_k$.

On carrying out the necessary calculations, we get

$$E_0 = \frac{p_0 R_0}{(\gamma - 1)^2}(\mu^{-(\gamma-1)/2} - 1)[(3\gamma - 1)\mu^{-(\gamma-1)/2} - \gamma - 1] \tag{3.5}$$

$$E_s = \frac{(\gamma + 1)p_0 R_0}{(\gamma - 1)^2}(1 - \lambda^{-(\gamma-1)/(\gamma+1)})^2$$

$$\mu = \frac{s}{\gamma - 1}(\gamma + 1 - 2s^{(\gamma-1)/2}), \quad s = \frac{\gamma + 1}{\gamma - 1}\lambda^{2/(\gamma+1)} - \frac{2}{\gamma - 1}\lambda, \quad \lambda \in (0, 1) \tag{3.6}$$

It follows from (3.5) and (3.6) that, in the case of weak compression when $s \to 1$ ($\lambda \to 1$, $\mu \to 1$), $E_0 \to 0$, $E_s \to 0$, after some simplifications we get

$$\eta \sim \frac{\gamma - 1}{2}(1 - s^{-(\gamma-1)/2})^{-1} \tag{3.7}$$

for the ratio $\eta = E_s/E_0$.

In the case of high compression ($s \sim 0$)

$$\eta \sim \frac{\gamma + 1}{3\gamma - 1}\left(\frac{\gamma + 1}{\gamma - 1}\right)^{2(\gamma-1)} \tag{3.8}$$

Hence, the gain in the case of weak compression $s \to 0$ is very large ($\eta \to \infty$) while, in the case of high compression, it is finite and depends on γ. In particular, when $\gamma = 3$, $\eta \sim 64/11$. By formally directing γ to infinity, we obtain from (3.8) that $\eta \to \frac{1}{3}e^4$. Whereas a large gain is natural in the first case since, with rapid compression, only a small mass of the gas is brought into motion, in the case of high compressions, a situation where the gain is finite, but nevertheless may be quite appreciable, is not expected since, when $s \to 0$, the instant $t_+ \to t_k$ and, consequently, a gain in energy is attained by a uniform compression when $t > t_*$ with a very high velocity. We note that, in the general case, η is a monotonic function of s.

The curves $\eta(s)$ when $\gamma = 1.1, 1.4, 5/3, 2$ and 3 are shown in Fig. 3.

Remark 1. It would be tempting to achieve even greater gains by invoking some other more-general equations of state with a gradual increase in the pressure as a function of density. However, a number of calculations using relationships (2.6) and (2.7) and the optimality principle which has been formulated in the

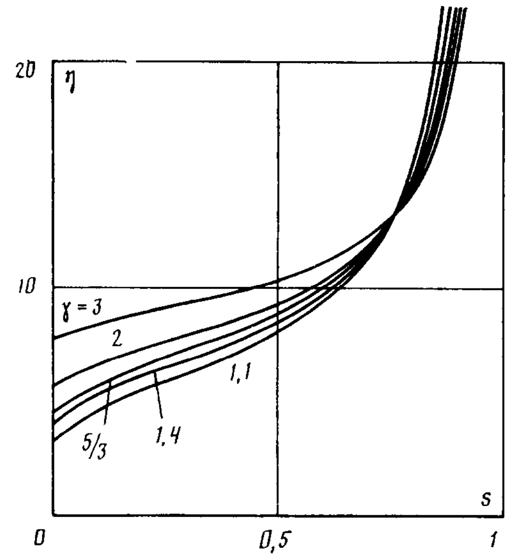

FIG. 3.

case of equations of state with an exponential dependence of the pressure on the density showed that it was not possible to achieve a gain greater than $\frac{1}{3}e^4$.

Remark 2. When $t - t_k$, a shock wave, with a constant velocity, starts to propagate from the rigid wall $\xi = R_f$ until it encounters the piston R_t at a certain instant of time T. It is clear that, up until this instant, it is advantageous to continue to move at a constant velocity v_* for the purpose of achieving an even greater degree of compression with a low energy expenditure.

4. Let us now consider the case of the compression of cylindrical and spherical layers of gas with a minimum expenditure of energy. Unlike the case of a planar layer, it is not possible to obtain an exact analytical solution of such problems. This is primarily due to the fact that the exact relationship between the velocity of the gas and the velocity of sound at the piston is unknown and this means that the functional $E(f)$ cannot be written explicitly. Furthermore, the analytical solution of problems on the determination of the laws of motion of cylindrical and spherical pistons, which ensure the unbounded shockless compression of layers, is unknown even in the self-similar cases of the compression of a cylinder and a sphere [3, 4].

On account of this, we shall next construct an approximate solution of the problem which has been formulated which rests on the following hypothesis: the velocity of sound and the velocity of the gas at the piston are related by the dependence (2.2), that is, one of the Riemann invariants keeps a constant value in the neighbourhood of the piston. Such a hypothesis is widely employed in the gas dynamics of one-dimensional flows [2] and frequently yields good quantitative results. In particular, it is clear that, at relatively small degrees of compression, such an approximation will be completely acceptable. Then, in the polytropic cylindrical case which we shall consider in greater detail, the functional $E(f)$ can be written in the form

$$E_c(f) = -2\pi p_0 \int_0^{t_k} \left(1 - \frac{\gamma - 1}{2} v\right)^{2\gamma/(\gamma-1)} fv\, dt, \quad v = f' \tag{4.1}$$

Here E_c is the work required per unit length of the generatrix of the cylinder.

We shall make use of the method of characteristic series [7–11] to construct a class of permissible controls V and to obtain an approximate determination of the flow in the region ACD. By analogy with the planar problem, we determine the coefficients of the series, for the case when the radii of the internal layer are nonzero, from the condition that all of the characteristics starting out from the line of the piston R_t are focused at the point (t_k, R_f). Here, if $R_f \neq 0$, the gas flow which arises will no longer be self-similar and dependent on the variable $\xi(t-t_k)^{-1}$.

We will introduce a new unknown function $\Gamma(u, t)$ (u is the velocity) such that [7]:

$$c^2 = (\gamma - 1)\left(\Gamma_t - \frac{1}{2}u^2\right), \quad \xi = \Gamma_u \tag{4.2}$$

The system of equations (2.1) can then be reduced to a single Monzha–Ampère equation

$$\Gamma_u[\Gamma_{uu}\Gamma_{tt} - (\Gamma_{ut} - u)^2] + N(\gamma - 1)\left(\Gamma_t - \frac{1}{2}u^2\right)(u\Gamma_{uu} + \Gamma_u) = 0 \tag{4.3}$$

It is assumed that $\Gamma_{uu} \neq 0$ so that the function $u(\xi, t)$ is implicitly defined by the second relationship of (4.2). The use of the variables u and t is convenient due to the fact that, in the neighbourhood of the point at which the characteristics are focused, the gradients of the gas dynamic quantities are large in the physical variables and the use of the variable u instead of ξ removes this difficulty. The line $u = 0$, which corresponds to a weak break in AD (Fig. 1), is a characteristic in the case of Eq. (4.3) ($c = 1$ along it).

We will represent the solution of Eq. (4.3) in the region ACD (Fig. 1) by the characteristic series

$$\Gamma = \sum_{k=0}^{\infty} a_k(t)\, u^k, \quad a_0 = (\gamma - 1)^{-1} t, \quad a_1 = R_0 - t \tag{4.4}$$

The coefficients $a_k(t)$ of series (4.4) [7] are determined by the successive integration of first-order ordinary differential equations. In order that the characteristics should be focused at the point A and that the velocity at this point should not be indeterminate (it depends on the angle of inclination of a given characteristic with the axis at point A), it follows from the second relationship of (4.2) that

$$a_k(t_k) = 0, \quad k = 2, 3, \ldots \tag{4.5}$$

Conditions (4.5) enable one to determine all of the arbitrary constants upon integrating the equations for $a_k(t)$, $k \geqslant 2$.

By calculating $a_2(t)$ and $a_3(t)$ and taking the corresponding segment of the series (4.4), we obtain the approximate relationship

$$\begin{aligned}\xi = R_0 - t - [R_0 - t - R_f^{1/2}(R_0 - t)^{1/2}](\gamma + 1)u + \frac{\gamma + 1}{2} \times \\ \times \left\{\frac{R_0 - t}{8}[19\gamma + 43 + 4(\gamma + 4)\ln R_f] - \frac{\gamma + 4}{2}(R_0 - t) + \frac{11}{8}(\gamma + 1)R_f -_t \right. \\ \left. - \frac{15\gamma + 27}{4} R_f^{1/2}(R_0 - t)^{1/2}\right\} u^2\end{aligned} \tag{4.6}$$

In order to find the law of motion $f(t)$ of the piston R approximately, we must put $\xi = f(t)$ and $u = f'(t)$ in (4.6). By integrating the resulting differential equation for $f(t)$ with the initial condition $f(0) = R_0$, it is possible, in particular, to obtain the Taylor coefficients of the expansion for $\eta(t)$

$$f'(t) = \eta(t) = \sum_{k=1}^{\infty} q_k t^k, \quad q_1 = -(\gamma+1)^{-1} R_0^{-1/2} (R_0^{1/2} - R_f^{1/2}) \tag{4.7}$$

The local convergence of series of the type (4.4) for small u and t has been established [11]. However, a number of applications of series (4.4), in particular, in the case of problems concerned with the efflux of a gas into a vacuum [12] has shown that the domain of their convergence (which is at the same time often very very rapid) may be quite large and also include, for example, the boundary adjacent to the vacuum. In order to obtain solutions in the case of high velocities in the part of the domain ACD which is adjacent to the piston, it is advisable to employ the characteristic series directly in the physical variables ξ, t for functions u and c of the form

$$c = \sum_{k=0}^{\infty} b_k(t)(\xi + t - R_0)^k \tag{4.8}$$

Here, splicing of series of series of the type of (4.4) with series of the type of (4.8) can be employed as well as various methods for speeding up their convergence such as, for example, by using Padé approximations [12]. We shall therefore assume that the function $\eta(t)$ ($\eta(t) \leqslant 0$, $\eta(0) = 0$), which accomplishes the motions of the piston with the focusing of all of the characteristics at point A, is approximately determined. We note that, in the case where $R_f = 0$ when a cylinder or a sphere is compressed, a representation of $\eta(t)$ of the form of (4.7) which is analytic with respect to t is impossible. Terms with logarithmic singularities appear in the expansions and $\eta(t)$ can be found either numerically [7] or by the use of the technique of expansions from [13].

5. Let us now consider the problem of constructing the optimal control $v(t)$ in the class of permissible controls $V\{\eta(t) \leqslant v(t) \leqslant 0\}$ which minimizes the functional (4.1). Using Pontryagin's maximum principle, the problem reduces to minimizing the function

$$\min_{v \in V} \left\{ -2\pi p_0 v f \left(1 - \frac{\gamma-1}{2} v\right)^{2\gamma/(\gamma-1)} - q(t) v \right\} \tag{5.1}$$

with the condition (the Euler condition)

$$q'(t) = -2\pi p_0 v \left(1 - \frac{\gamma-1}{2} v\right)^{2\gamma/(\gamma-1)} \geqslant 0$$

and the supplementary conditions of traversality on $q(t)$, an auxiliary function, such that the trajectory of the piston passes through the point (R_k, t_k).

The nature of the change in the function being minimized, $W(v, t)$ in (5.1) is the same as that in the case of the function shown in Fig. 2. Here

$$W_v(v, 0) = -2\pi p_0 R_0 - \lambda_1 > 0, \quad q(0) = \lambda_1 < 0$$
$$q(t_k) = -\lambda_2 < 0$$

Hence, as in the planar case, up to a certain instant t_* which is determined from the conditions

$$-2\pi p_0 \eta(t_*) \left(1 - \frac{\gamma-1}{2} v_*\right)^{(\gamma+1)/(\gamma-1)} \left(1 - \frac{3\gamma-1}{2} v_*\right) - q(t_*) = 0 \tag{5.2}$$
$$v_* = \eta(t_*)$$

the optimal control is identical to the law $\eta(t)$ when the characteristics are focused at point A

$$v(t) = \eta(t), \quad 0 \leqslant t \leqslant t_* \tag{5.3}$$
$$q(t) = -2\pi p_0 \int_0^t \eta(t)\left(1 - \frac{\gamma-1}{2}\eta(t)\right)^{2\gamma/(\gamma-1)} dt + \lambda_1$$

and the function W is a minimum when $v = v_*$.

When $t > t_*$, we get the following system of equations which determine the optimal control

$$\left(1 - \frac{\gamma-1}{2} v\right)^{(\gamma+1)/(\gamma-1)}\left(1 - \frac{3\gamma-1}{2} v\right) = -\frac{1}{2\pi p_0}\frac{q(t)}{f(t)} \tag{5.4}$$

$$q' = -2\pi p_0 v\left(1 - \frac{\gamma-1}{2} v\right)^{2\gamma/(\gamma-1)}, \quad f' = v \tag{5.5}$$

By eliminating $q(t)$ from (5.5) using (5.4), we obtain the relationship

$$-v^2\left(1 - \frac{\gamma-1}{2} v\right) = fv'\left(2 - \frac{3\gamma-1}{2} v\right) \tag{5.6}$$

Next, by eliminating the function $f(t)$ from (5.6) using (5.5), we get a second-order equation for $v(t)$, the general integral of which we write in the form (C_1 and C_2 are arbitrary constants)

$$C_1 t + C_2 = \int_{v_*}^{v} \frac{(2 - (3\gamma-1)/2\eta)\, d\eta}{\eta^4 (1 - {}^1\!/_2(\gamma-1)\eta)^{2\gamma/(\gamma-1)}} = \Lambda(v) \tag{5.7}$$

Here, the relationships

$$C_1 v^2 f\left(1 - \frac{\gamma-1}{2} v\right)^{(\gamma+1)/(\gamma-1)} = -1$$
$$R_* = f(t_*) > 0, \quad v_* = v(t_*) < 0, \quad v_k = v(t_k) \tag{5.8}$$

$$C_1 t_k + C_2 = \Lambda(v_k), \quad C_1 t_* + C_2 = \Lambda(v_*) = 0 \tag{5.9}$$

$$C_1(t_k - t_*) = \Lambda(v_k) < 0, \qquad \frac{v_k^2 (1 - {}^1\!/_2(\gamma-1)v_k)^{(\gamma+1)/(\gamma-1)}}{v_*^2 (1 - {}^1\!/_2(\gamma-1)v_*)^{(\gamma+1)/(\gamma-1)}} = \frac{R_*}{R_k} > 1 \tag{5.10}$$

are satisfied.

It can be shown that an instant when there is a change over $t_* \in (0, t_+)$ is always found. Actually, we get an equation for finding t_* from (5.7)–(5.10)

$$F(t_*) = v_*^2 f(t_*)\left(1 - \frac{\gamma-1}{2} v_*\right)^{(\gamma+1)/(\gamma-1)} + \frac{t_k - t_*}{\Lambda(v_k(t_*))} = 0 \tag{5.11}$$

where the function $v_k(t_*)$ is implicitly determined from (5.10).

It follows from (5.10) that $|v_k(t_*)| > v_*|$, that is $\Lambda(v_k(t_*)) < 0$ always. When the values of t_* are close to zero, v_* and $v_k(t_*)$ are also close to zero. For such t_*, it follows from (5.7) that

$$\Lambda(v_k(t_*)) \sim -\frac{2}{3}(v_k^{-3}(t_*) - v_*^{-3}) \sim -\frac{2}{3} v_*^{-3}\left[\left(\frac{R_k}{R_0}\right)^{3/2} - 1\right] < 0$$

But from Eq. (5.11) we get

$$F(t_*) \sim v_*^2 R_0 + O(v_*^3) > 0$$

$$F(t_*) \sim v_*^2 R_0 + O(v_*^3) > 0$$

On the other hand, when $t_* \to t_+ - 0$ (Fig. 1), we shall have

$$v_* \to \eta(t_+),\ \Lambda(v_k(t_*)) \to -0,\ t_k - t_+ > 0$$

and we therefore get from (5.11) that $F(t_*) \to -\infty$. $F(t_*)$ is a continuous function of t_* and, consequently, a root of $F(t_*)$ is always found in the interval $(0, t_+)$.

Finally, we write the law of optimal control when $t_* \leqslant t < t_k$ in parametric form as

$$\xi = A\mu^{-2}\left(1 - \frac{\gamma - 1}{2}\mu\right)^{-(\gamma+1)/(\gamma-1)}, \quad \mu \in [v_*, \mu_k]$$
$$t = -A\int_{v_*}^{\mu} \mu^{-4}\left(2 - \frac{3\gamma - 1}{2}\mu\right)\left(1 - \frac{\gamma - 1}{2}\mu\right)^{2\gamma/(\gamma-1)} d\mu + t_* \tag{5.12}$$

where the constants t_*, A and μ_k are determined from the condition that points B and D are traversed by the piston and μ is a parameter.

It is important to point out that the law of optimal control (5.12) is universal and independent of the actual form of the function $\eta(t)$ by means of which the control of the initial stage in the motion of the piston is realized according to Eq. (5.3) prior to the instant of change over $t = t_*$.

Calculations of the optimal control law using the formulae constructed in Secs 3 and 4 are shown in Fig. 4 in the case of the compression of a cylindrical layer ($R_f = 0.4$, $R_0 = 1$) of a gas with $\gamma = 1.4$. It was found that the final segment CB of the piston trajectory, calculated using (5.12), is practically linear and, here, $t_* = 0.361$ and $R_* = 0.918$. A calculation of the flow field in the curvilinear triangle ABC (or directly over the whole of the trangle ABD) can be carried out by the method of characteristics, for example. In this case, it is necessary to solve a Cauchy problem with approximate data on a known curvilinear characteristic DCA (or DA) and with no passage conditions on a known trajectory of the piston CB (after points of change over of the control) or BCD.

In the spherical case, arguments can be used which are completely analogous to the case of axial symmetry. Instead of the functional (4.1) for the energy of the piston $E_s(f)$, we shall have

$$E_s(f) = -4\pi p_0 \int_0^{t_k} \left(1 - \frac{\gamma - 1}{2} v\right)^{2\gamma/(\gamma-1)} f^2 v\, dt$$

The problem of the optimal control of the piston can also be completely solved here in

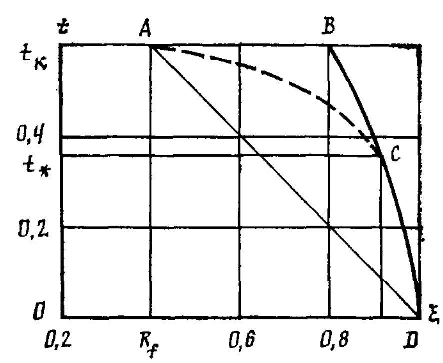

FIG. 4.

quadratures although the control law will differ substantially from (5.3), (5.12). Proof of the existence of a control changeover point t_* is also far more difficult. The final formulae for the optimal control laws in the spherical case are presented in [1].

REFERENCES

1. SIDOROV A. F., On the optimal shockless compression of gas layers. *Dokl. Akad. Nauk SSSR* **313,** 2, 283–287, 1990.
2. STANYUKOVICH K. P., *Non-Steady State Motions of a Continuous Medium*. Gostekhizdat, Moscow, 1955.
3. ZABABAKHIN E. I. and ZABABAKHIN I. E., *Unbounded Cumulation Phenomena*. Nauka, Moscow, 1988.
4. NACKOLLS J. G., Realization of inertial thermonuclear synthesis. *Uspekhi Fiz. Nauk* **143,** 3, 467–482, 1984.
5. SEDOV L. I., Methods of Similarity and Dimensionality in Mechanics. Gostekhizdat, Moscow, 1957.
6. ALEKSEYEV V. M., TIKHOMIROV V. M. and FOMIN S. V., *Optimal Control*. Nauka, Moscow, 1979.
7. SIDOROV A. F., A method of solving certain boundary value problems for non-linear equations of the hyperbolic type and the propagation of weak shock waves. *Prikl. Mat. Mekh.* **36,** 3, 426–434, 1972.
8. ZUBOV E. N. and SIDOROV A. F., On the solution of a boundary value problem for a non-steady state spatial gas flow and the propagation of weak spherical shock waves. *Chisl. Metody Mekhaniki Sploshnoi Sredy, Vychisl. Tsentr Sib. Otd. Akad. Nauk SSSR, Novosibirsk* **3,** 3, 32–50, 1972.
9. SIDOROV A. F., An approximate method for solving certain problems concerning the spatial efflux of a gas into a vacuum. *Chisl. Metody Mekhaniki Sploshnoi Sredy, Vychisl. Tsentr Sib. Otd. Akad. Nauk SSSR, Novosibirsk* **7,** 5, 137–148, 1976.
10. SIDOROV A. F., Application of characteristic series to the solution of three-dimensional problems in gas dynamics. In *Numerical Methods in Fluid Dynamics*. Mir, Moscow, 1984.
11. BAUTIN S. P., Analytical solutions of a problem on the motion of a piston. *Chisl. Metody Mekhaniki Sploshnoi Sredy, Vychisl. Tsentr Sib. Otd. Akad. Nauk SSSR* **4,** 1, 3–15, 1973.
12. BASHKIRTSEVA I. A., Application of numerical–analytical methods to problems concerning the efflux of a gas into a vacuum. Questions of atomic science and technology. *Ser. Mathematical Modelling of Physical Processes*. TsNIIatominform., Moscow, **3,** 84–89, 1990.
13. VERSHININ S. V. and SIDOROV A. F., On the behaviour of the solutions of double wave equations in the neighbourhood of a quiescent domain. *Prikl. Mat. Mekh.* **39,** 6, 1043–1050, 1975.

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